71

PHASE FOLLOWING BEHAVIOUR OF AN AUTOMATIC PHASE CONTROL CIRCUIT WITH RESPECT TO A SIGNAL IN PRESENCE OF A RANDOM NOISE*†

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(*Received July 8, 1965, Resubmitted May 30, 1966*)

ABSTRACT. In this paper the response of an automatic phase control circuit to several different classes of signals has been studied in the presence of noise. The effects of variation of the equivalent gain parameter of the phase locking loop on the system performance for various values of the input signal to noise ratio have also been studied. The concepts of loop noise bandwidth and root-mean-square phase error have been briefly reviewed with particular reference to a band limited noise. Experimental results regarding the performance of the APC circuit in relation to the receiption of a FM signal have been presented and found to be in good agreement with the results of the analysis.

A = amplitude of the incoming signal.
ω_2 = angular frequency of the free running local oscillator.
ω_1 = angular frequency of the incoming signal.
Ω = open loop frequency error in angular measure.
$X(t)$, $Y(t)$ = uncorrelated Gaussian variable of angular bandwidth ω_t.
τ = correlation time.
σ_i^2 = variance of the input noise.
m_i = index of modulation at the input.
m_0 = index of modulation at the output.
$\epsilon(t)$ = phase modulation of the VCO due to noise.
β = sensitivity of the VCO.
K = maximum possible synchronisation range in angular measure.
σ^2 = variance of the noise at the input to phase detector.
σ_φ^2 = variance of the noise at the output of the phase detector.
B_L = loop noise-bandwidth.
μ_S = equivalent linearised gain of the phase detector for the singal.
μ_N = equivalent linearised gain of the phase detector for the noise.

INTRODUCTION

The automatic phase control circuit is essentially an oscillator, the phase of which is locked to an input reference oscillation. It is a narrow-band feedback

*This work was done at the Institute of Radio Physics and Electronics, Calcutta.
†Part of this paper was presented at the "Symposium on Telecommunication and Electronics, Feb. 26-27, 1963 held at the Institute of Radio Physics and Electronics, Calcutta entitled as "On the Performance of an APC Circuit" (unpublished).

device and consists of a phase detector, a linear filter and a voltage controlled oscillator. The analysis of such a system, even if it is noise-free, is rather difficult because of the inclusion of an error sensing device which is a sinusoidal function of the error itself. But it simplifies considerably if the phase error is small because then the behaviour of the loop can be predicted from a linear analysis of the loop. In such a case it is known that the system can be made to synchronise with respect to the reference input if the open loop frequency error lies within the limits of synchronisation range. But somtimes it has been found that the system may not be synchronised although well within the synchronisation range. This is because of the fact that the 'pull-in" range is different from the "pull-out" range and this requires a non-linear analysis of the loop to determine almost exactly and completely the performance characteristics of the loop It is to be further noted that because of the narrow-band feedback process it reduces internally generated noises as well as uncontrolled disturbances that may accompany the input signal to the system. The present purpose of this paper is to develope an analytical method that will help us to know the signal handling capacity of the system as well as to evaluate the output SNR of the system in terms of the input SNR. This, in turn, can be used to find the threshold criterion of the loop The response of such a circuit to a signal contaminated with stationary random noise has been studied by many authors (Viterbi, A. J. 1963, *et al.*). A convenient approach in such studies is to replace the sinusoidal non-linearity of the phase detector by a linear one whose gain is the quivalent gain of the device, applying quasilinearisation techniques. The other approach employs Fokker-Plank or continuous random walk techniques to find the statistics of random process.

In section 2 a general method for studying the response of an APC circuit to a FM signal contaminated with stationary random noise has been developed. The approach utilised here is to find out the equivalent linearised gains of the phase detector for the signal and noise separately and once the values of the equivalent gains μ_S and μ_N are known the non-linear system can be analysed as two linear systems—one containing the parameter μ_S and the other incorporating the parameter μ_N. However, it is to be noted that since the system is a nonlinear one the value of μ_S and μ_N do depend on the strengths of the signal and noise at the input to the phase detector. Expressions for the equivalent linearised gain of a general type of non-linear element have been developed

The response of the APC circuit to a FM signal only has been studied in section 3 utilising the concept developed in section 2. An expression for the maximum permissible input modulation index in terms of the modulating frequency, filter parameters and maximum value of the locking range has been developed.

Section 4 deals with the response of the APC circuit to a continuous wave signal which is contaminated with stationary random noise. It also deals with the evaluation of the variance of the noise or the m.s. phase error at the input to the phase detector when the APC loop is closed through a low-pass filter This is followed by a discussion in section 5 of the effect of noise on the locking behaviour of the APC circuit. The probability that the system may fall out of lock has been calculated and depicted graphically for different values of the input carrier to noise ratio.

The response of the APC circuit to a FM signal contaminated with stationary random noise has been studied in section 6, particularly when the carrier is in tune with the centre frequency of the voltage controlled oscillator. Due to the difficulty in the analytical computation of the output SNR, in terms of the input SNR, a graphical method for the evaluation of such an expression has been described. The effect of variation of the gain parameter on the performance of such a circuit has also been studied in this section.

In section 7 the concepts of loop noise bandwidth and r.m.s. phase error have been briefly reviewed with particular reference to a bandlimited noise which is common inpractice. This is followed by a description of the experimental results with regard to the response of the APC circuit to a FM signal and noise in section 8 These are in good agreement with the results of the analysis presented in the text

Response of an APC circuit to signal contaminated with stationary random noise :

In this section a general method of analysing the response of an automatic phase control circuit to frequency modulated signal that is contaminated with a stationary random noise will be developed. Here the cases of interest are (i) when the centre frequency of the voltage controlled oscillator (VCO) is in tune with the incoming signal and (ii) when the centre frequency of the VCO is slightly out of tune with the incoming signal but the difference of frequency between them is not so high as to cause the system to fall out of lock during the phase following of the modulation cycle by the VCO In this case, i e., when the input to the system consists of signal and noise, the presence of the noise will cause a phase jitter at the output of the VCO over and above the so-called slow variation of the output phase of the VCO due to the frequency modulation of the input signal. The amount of phase jitter will depend upon the close loop noise bandwidth, locking range of the system and the amount of initial detuning of the VCO from the incoming signal These parameters again, on the otherhand, will limit the maximum permissible value of the input modulation index. If the input signal to noise ratio (SNR) is high the amount of phase jitter at the output will almost be the same as that at the input if the noise power be taken equal to that in the close-loop noise bandwidth. This can be found from a straight forward feedback theory. If, however, the noise power is comparable to the signal power, the noise will also cause a change in the equivalent gain of the phase-detector, which is essentially a nonlinear device. The equivalent gain for the noise, in turn, will be affected by the value of signal power.

The nonlinear characteristic exhibited by a phase detector is not a very usual one in the sense that the output is a sinusoidal function of the input. Analysis of such a system is rather difficult and one has to take resort to appropriate approximations. A convenient approach is to replace the sinusoidal nonlinearity by a linear one whose gain is the expected gain of the actual device applying essentially Booton's quasi-linearisation technique

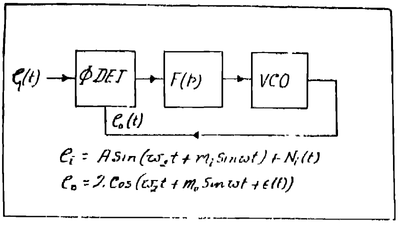

Fig 1 Block diagram of a typical automatic phase control circuit The input to the system consists of a FM signal contaminated with a stationary random noise

Let us consider the automatic phase control circuit as depicted in Fig 1 The input is a frequency modulated signal which is accompanied by a stationary random noise The noise possesses a Gaussian amplitude probability distribution with a mean zero and variance σ^2 and has power spectral density which is equivalent to that obtained by passing 'white' Gaussian noise through a single tuned IF filter having the centre frequency equal to that of the VCO and angular bandwidth ω_i The net input can, therefore, be written as

$$e(t) = A\sin(w_1 t + m_i \sin wt) + X(t)\sin\omega_2 t + Y(t)\cos\omega_2 t, \tag{2.1}$$

where $X(t)$ and $Y(t)$ are uncorrelated Gaussian variable of angular bandwidth The auto-correlation function of $X(t)$ and $Y(t)$ can be assumed to be given by

$$R_i[X(t)] = R_i[Y(t)] = \sigma_i^2 \exp(-\omega_i|\tau|), \tag{2.2}$$

where τ is the correlation time. Corresponding to this autocorrelation function the power spectral density of the input can be written as

$$G_i(\omega) = 4\sigma_i^2 \frac{\omega_i}{\omega^2+\omega_i^2} \tag{2.3}$$

Assumihg the output of the voltage controlled oscillator to be of the

$$e_0(t) = 2\cos[\omega_2 t + m_0 \sin\omega t + \epsilon(t)], \tag{2.4}$$

where m_0 is the modulation index at the output and $\epsilon(t)$ is the phase modulation of the VCO due to the noise. Therefore the governing equation of the loop is given by (see Appendix A)

$$\frac{d\phi}{dt} = \Omega + \frac{d}{dt}(m_i \sin\omega t) - \beta F(P)[A \sin\phi + N(t)], \quad \ldots \quad (2.5)$$

where

$$\begin{aligned} \phi &= \omega_1 - \omega_2 + (m_i - m_0)\sin\omega t - \epsilon(t) \\ \Omega &= \omega_1 - \omega_2 \\ N(t) &= -X(t)\sin[m_0 \sin\omega t + c(t)] \\ &\quad + Y(t)\cos[m_0 \sin\omega t + c(t)] \end{aligned}$$

and β is the sensitivity of the VCO. It can be shown that $N(t)$ is a stationary process process with exactly the same autocorrelation function as $X(t)$ or $Y(t)$ (Viterbi, 1963), Eq. (2 5) suggests an analytical equivalent of the APC circuit that is shown in Fig 2 For the in-tune carrier, the corresponding loop equation is given by

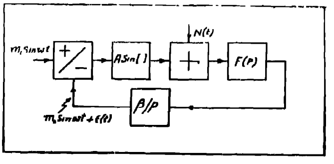

Fig 2 Equivalent analytical representation of the automatic phase control circuit of Fig. 1.

$$\frac{d\phi}{dt} = \frac{d}{dt}(m \sin wt) - \beta F(P)[A \sin\phi + N(t)] \quad \ldots \quad (2.6)$$

where $\quad \phi = (m_i - m_0)\sin wt - \epsilon(t) \quad (2.7)$

As stated earlier ih this case a convenient approach, although approximate, is to find out the equivalent linearised gains of the phase detector for the signal and the noise separately and to break the signal loop into two equivalent analytical loopsone for the signal and the other for the noise (see—Fig. 3d) for studying the signal and noise respeonse of the APC circuit. Now when the system is in lock it was seen that the instantaneous loop error consists of a signal error, a noise error and steady state phase difference that depends on the locking ratio. The noise at the input to the phase detector will again be assumed to a stationary random noise the magnitude of which is normally distributed with a mean zero and variance σ^2. Specifically assuming that the nonlinearity can be represented by

$$F_+(\phi) = \Sigma\, a_n(\phi)^{1/n} \quad \text{for} \quad \phi > 0 \quad \ldots \quad (2.8)$$

$$F_-(\phi) = \Sigma\, b_n(-\phi)^{1/n} \quad \text{for} \quad \phi < 0 \quad \ldots \quad (2.9)$$

One can show that the equivalent linearised gains of the phase detector for signal and noise are respectively given by (Sawaragi and Sugai, 1959).

$$\mu_S/A = \frac{2}{M}\left[\frac{1}{2\pi}\int_{c+} F_+(jw)I_1(jwM)\exp\left(-\tfrac{1}{2}\sigma^2\omega^2\right)d\omega\right.$$

$$\left.+\frac{1}{2\pi}\int_{c-} F_-(jw)I_1(jwM)\exp\left(-\tfrac{1}{2}\sigma^2\omega^2\right)d\omega\right] \quad \ldots \quad (2.10)$$

$$\mu_N/A = \frac{1}{2}\int_{c+} j\omega F_+(j\omega)\, J_0(j\omega M)\exp\left(-\tfrac{1}{2}\sigma^2\omega^2\right)d\omega$$

$$+\frac{1}{2}\int_{c-} j\omega F_-(jw)\, J_0(jwM)\exp\left(-\tfrac{1}{2}\sigma^2\omega^2\right)dw \quad \ldots \quad (2.11)$$

where M is the modulation error C_+ and C_- are integral paths along the straight lines from $-j\delta-\infty$ to $-j\delta+\infty$ and from $j\gamma-\infty$ to $j\gamma+\infty$ respectively. $I_1(x)$ is the modified Bessel function of order one and argument x, $F_+(j\omega)$ and $F_-(j\omega)$ are respectively the Fourier transform of the nonlinearity when $\phi > 0$ and $\phi < 0$ and they are given by

$$F_+(j\omega) = \sum_n a_n \frac{\Gamma\left(\frac{1}{n}+1\right)}{(j\omega)^{1+1/n}} \quad \ldots \quad (2.11a)$$

$$F_-(jw) = \sum b_n \frac{\Gamma\left(\frac{1}{n}+1\right)}{(j\omega)^{1+1/n}} \quad \ldots \quad (2.11b)$$

At this point one can physically argue that the equivalent linearised gain of the phase detector with respect to the signal in the off-tune case will be smaller than in the in-tune case. Therefore, the probability of loss of lock per cycle for a signal with low modulation index in the off-tuned case will be larger than in the in-tune case.

In the sections to follow we shall discuss in detail the performance of the APC circuit particularly when the incoming signal is in tune with the voltage controlled oscillator.

Response of a Frequency Modulated Signal Only.

A method based on the principle of quasilinearisation technique for analysing the response of the APC circuit to an FM signal is presented here. The method is limited to the case of in-tune carrier only. In this case if the locking range, system bandwidth and the modulation index of the input signal are properly adjusted, it is reasonable to assume that modulation error as well as the distortion components at the output of the VCO will be small. Therefore the equivalent linearised gain of the phase detector isgiven by

$$\mu_S = \frac{A}{M} \frac{1}{\pi} \int_0^{2\pi} \sin (M \sin \omega t) d\omega t$$

$$= 2A \frac{J_1(M)}{M} \quad \cdots \quad (3.1)$$

where A is the signal strength $J_1(M)$ is the Bessel function of order one and index M and M is the modulation error. From the equivalent analytical loop for the signal (Fig. 3a) it is easy to show that

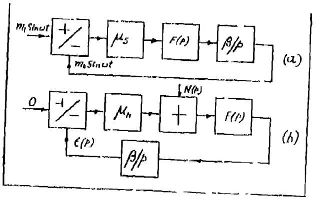

Fig 3(a) Equivalent analytical representation of the automatic phase control circuit of Fig. 1 for the signal component only The sinusoidal nonlinearity of the phase detector has been replaced by a linear one whose gain for the signal (μ_S) is the expected gain of the device itself.

Fig. 3(b) Equivalent analytical representation of the automatic phase control circuit of Fig. 1 for the noise component only. The sinusoidal nonlinearity of the phase detector has been replaced by a linear one whose gain for the noise (μ_N) is the expected gain of the device itself.

$$\frac{M}{m_i} = \frac{1}{1+2K \frac{J_1(M)}{M} \frac{F(P)}{P}} \quad \cdots \quad (3.2)$$

Thus knowing the value of the modulation error M for a definite value of the input modulation index it is easy to find out a relation between the input modulation index and the output modulation index (m_0) in terms of the loop parameters.

The maximum permissible value of the input modulation index can be found from Eq. (3.2) by taking M to be nearly equal to $\pi/2$ and one can easily snhow that

$$[m_i]_{max} \simeq \pi/2 \left| \left[1+0.7K \frac{F(P)}{p} \right] \right| \quad \cdots \quad (3.3)$$

It is important to note that the above analysis, although not very accurate as it does not take into account distortion components, gives an idea about the nature of variation of the maximum permissible value of the input modulation index with modulating frequency. The analysis given above can, however, be extended to the slightly off-tuned carrier depending upon the locking range of the system. In this case it is only to be remembered that the maximum permissible input modulation index will be smaller than that of the in-tune carrier, the amount of which is essentially dependent on the locking ratio and the type of filter used.

Response of an APC Circuit to a CW Signal Contaminated with Stationary Random Noise .

In this section a study will be made of the response of the automatic phase control circuit to a CW signal contaminated with stationary random input time function. The amount of phase filter at the output of the VCO caused by the input and the amount of initial detuning of the VCO from the incoming CW signal

The input to the system consists of the signal and random stationary noise, the properties of which have been described elsewhere in the text. Let us assume that the output of the VCO is of the form

$$e_0(t) = 2\cos[\omega_2 t + \epsilon(t)] \tag{4.1}$$

where $\epsilon(t)$ represents the phase jitter of the VCO output introduced by the noise It is easy to show that the governing equation of the loop is given by,

$$\frac{d}{dt}[\epsilon(t)] = KF(P)\sin\epsilon(t) \tag{4.2}$$

when the noise-phase variation is such that $\epsilon(t)$ remains within the stretch between $-\pi/2$ and $+\pi/2$, one can approximate the sinusoidal nonlinearity by the following relation

$$\sin\phi = F_+(\phi)+F_-(\phi) \tag{4.3}$$

where

$$F_+(\phi) = .04(\phi)+.71(\phi)^{2/3} \text{ for } \phi \geq 0 \tag{4.4}$$

$$F_-(\phi) = .04(-\phi)+.71(-\phi)^{2/3} \text{ for } \phi < 0 \tag{4.5}$$

It is to be noted that if $\epsilon(t)$ goes beyond the stretch from $-\pi/2$ to $+\pi/2$ then one has to find out a suitable relation to approximate the sinusoidal non-linearity Therefore with this approximation and remembering that

$$|F_+(\phi)| = |F_-(\phi)| \tag{4.6}$$

one can easily show from Eq. (2.11) that the equivalent linearised gain of the phase detector with respect to the noise is given by

$$\mu_N/A = \frac{1}{\pi}\int_{c+} j\Lambda \quad F_+(j\omega)\exp \quad \sigma^2\omega^2\Big) d\omega \tag{4.7}$$

$$= 0.04+0.32(\sigma)^- \tag{4.7a}$$

where σ^2 is the variance of the noise at the input to the phase detector. The variation of the equivalent linearised gain of the phase detector for noise is shown in Fig. 4. From the equivalent analytical loop of Fig. 3(*b*) it is easy to show that

$$\sigma^2 = \frac{1}{4\pi A^2}\int_{-\infty}^{+\infty}\left|\frac{KF(j\Lambda)}{j\omega+K[.04+.32(\sigma)^{-1/3}F(jw)}\right| G_i(w)dw, \qquad \ldots \quad (4.8)$$

and

$$\sigma_\varphi^2 = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\left|\frac{KF(jw)}{jw+K[.04+.32(\sigma)^{-1/3}\,F(jw)}\right| G_i(w)d\omega \qquad \ldots \quad (4.9)$$

where
$$K = A\beta$$

and σ_φ^2 is the variance of the noise at the output of the phase detector. From above equations it is possible to find the values of σ^2 and σ_φ^2 in terms of the pnput vari-

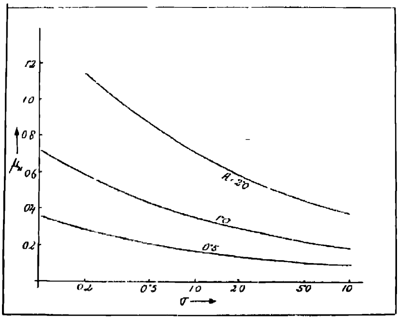

Fig. 4. Variation of the equivalent linearised gain (μ_N) of the phase detector for the noise in presence of a CQW-signal of amplitude A.

ance and loop parameters. For reasons of difficulty in analytical computation of the above parameters it is sometimes advisable to take resort to graphical method of computation.

For example, taking the case of an APC circuit with a low pass filter having the transfer function given by

$$F(jw) = \frac{1}{1+j\omega T} \qquad \ldots \quad (4.10)$$

and comparing Eq. (4.8), (4.9) (4.10) and (2.3) one can easily show that

$$\sigma^2 \cdot \frac{A^2}{\sigma_i^2} = \frac{K}{.04+.32(\sigma)^{-1/3}} \cdot \frac{1+\omega_i T}{K[.04+.32(\sigma)^{-1/3}]+w_i(1+w_i T)} \qquad \ldots \quad (4.11)$$

$$\frac{\sigma_\varphi^2}{\sigma_i^2} = \frac{(1+w_i T)[.04+.32(\sigma)^{-1/3}]}{[.04+.32(\sigma)^{-1/3}]+\dfrac{w_i(1+w_i T)}{K}} \quad \ldots \quad (4.12)$$

From the plot of Fig. 5 one can easily find the value of σ^2 for a particular loop parameter and hence knowing σ^2 it is easy to find the value of σ_φ^2 from Eq (4.10).

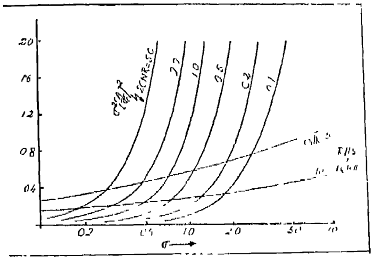

Fig. 5. Illustrates a graphical method for evaluating the value of the variance of the noise (σ) at the input to the phase detector in terms of the variance of the noise at the input to the system (vide Eq 4.11).

Effect of noise on the locking behaviour of an automatic phase control circuit:

If the input to an automatic phase control circuit is corrupted with band-limited white Gaussian noise with mean zero and variance σ_i^2, there will be phase jitter in the output of the voltage controlled oscillator. The amount of phase jitter will, however, depend upon the input signal to noise ratio and the system bandwidth. To understand in physical terms how noise affects the locking behaviour of an APC circuit one may study the nature of the instantaneous variations in phase-difference ϕ between the reference oscillation and the local oscillation. When the input SNR is high and the detuning is small compared to the locking range, the amount of phase jitter will not be large enough to make the system fall out of lock.

If, however, the input SNR is low there are chances of losing lock of the system which will essentially depend upon the locking ratio (Ω/K), system bandwidth and input SNR. In order to investigate the case we will have to study the probability of the phase difference ϕ at the input to sine type nonlinearity of the APC circuit exceeding the phase stretch between $-\pi/2$ to $+\pi/2$ i.e. for the phase to be in the unstable region. This means that one has to study cumulative probabilities

$$P_1 = \text{Prob}\,(+\pi/2-\phi_0 < \phi < \pi) = \int_{+\pi/2-\phi_0}^{\pi} P(\phi)d\phi, \qquad (5.2)$$

and

$$P_2 = \text{Prob.}\,(-\pi < -\phi < -\pi/2-\phi_0) = \int_{-\pi}^{-\pi/2-\varphi} p(\phi)d\phi\ ; \qquad \ldots \quad (5.2)$$

where ϕ_0 is the steady state phase difference and $P(\phi)$ is the steady state distribution of phase at the input to the phase detector If, however, the carrier is in tune with the voltage controlled oscillator the above cuations reduce to the following simple forms :

$$P_1 = \int_{-\pi/2}^{\pi} P(\phi)d\phi \qquad \ldots \quad (5.3)$$

$$P_2 = \int_{-\pi}^{-\pi/2} P(\phi)d\phi \qquad \ldots \quad (5.3a)$$

It is to be noted that the relevant SNR is obviously not the input SNR (because of filtering action) but is related to it in a manner that depends upon the filter characteristics and the input SNR

The probability distribution of the phase when the carrier is in tune with the VCO calculated on the basis of Fokker-plank technique (Viterbi, 1963), for the first order toop is given by

$$P(\phi) = \frac{1}{2\pi I_0(\alpha)} \exp\,(\alpha \cos\phi) \qquad \ldots \quad (5.4)$$

where $I_0(\alpha)$ is the modified Bessel function of order zero and index α. Where

$$\alpha = 4A/K_1N \qquad \ldots \quad (5.5)$$

where $A^2/2$ is the power of the input carrier and N is the power spectral density of the input 'white' noise and K_1 is the of the sensitivity of the VCO. Expanding as

$$\exp\,(\alpha\cos\phi) = I_0(\alpha)+2\sum_n I_n(\alpha)\cos n\phi, \qquad \ldots \quad (5.6)$$

the values of P_1 and P_2 can be easily seen to be given by

$$P_1 = P_2 = \frac{1}{4} - \frac{1}{\pi I_0(\alpha)} \cdot \left[I_1(\alpha) - \frac{1}{3} I_3(\alpha) + \frac{1}{5} I_5(\alpha) \right] - \ldots \qquad \ldots \quad (5.7)$$

The plot of the cumulative probability P. is shown in Fig. 6 for different values of input carrier to noise ratio (CNR).

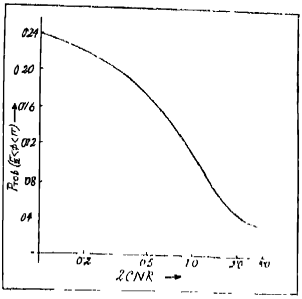

Fig 6 Shows the variation of the cumulative probability (P) with different values of the input carrier to noise ratio (CNR)

Response of an APC Circuit to a FM singlal contaminated with stationary random Noise

Here the response of an automatic phase control circuit to a FM singal that is contaminated with stationary random noise and is in tune with the centre frequency of the VCO will be considered The character of the noise has already has already been given elsewhere in the text Now in this case it is seen that the nonlinear element is a symmetrical one i e $|F_+(\phi)| = |F_-(\phi)|$ and therefore the expressions for the equivalent linearised gains of the phase detector can be written as (see Eq. (2.10) and (2.11)).

$$\frac{\mu_s}{A} = \frac{4}{M} \sum_n \frac{a_n \Gamma\left(\frac{1}{n}+1\right)\left(\frac{M^2}{2\sigma^2}\right)^{\frac{1}{2}}}{2\Gamma\left(1-\frac{1-1/n}{2}\right)\left(\frac{\sigma^2}{2}\right)^{\frac{1}{n}}} \; {}_1F_1\left(\frac{1-1/n}{2}, 2, -\frac{M^2}{2\sigma^2}\right) \quad \ldots \quad (6.1)$$

and

$$\frac{\mu_N}{A} = \sum_n \frac{a_n \Gamma\left(\frac{1}{n}+1\right)}{\Gamma\left(1-\frac{1-1/n}{2}\right)\left(\frac{\sigma^2}{2}\right)^{1/n}} \; {}_1F_1\left(\frac{1-1/n}{2}, 1, -\frac{M^2}{2\sigma^2}\right) \quad . \quad (6.2)$$

where ${}_1F_1(x, y, -z)$ is the confluent Hypergeometric function defined as

$$ {}_1F_1(x, y, -z) = 1 - \frac{x}{y} \cdot \frac{z}{1!} + \frac{x(x+1)}{y(y+1)} \cdot \frac{z^2}{2!} - \quad \ldots \quad (6.3)$$

and $\Gamma(x)$ is the Gamma function. Now in the case when the phase variation does not exceed the stretch from $-\pi/2$ to $\pi/2$ we can with resonable accuracy replace sinusoidal type of nonlinearity of the phase detector by Eq. (4.3) to Eq. (4.5) and therefore the above Eq. (6.2) and Eq. (6.3) reduce to the following simple forms

$$\mu_S = A\left[.04+.32(\sigma)^{-1/3}{}_1F_1\left(1/6,\, 2,\, -\frac{M^2}{2\sigma^2}\right)\right] \qquad \ldots \quad (6.4)$$

and

$$\mu_N = A\left[.04+.32(\sigma)^{-1/3}{}_1F_1\left(1/6,\, 1,\, -\frac{M^2}{2\sigma^2}\right)\right] \qquad (6.5)$$

The plots of the equivalent linearised gains of the phase detector for the signal and noise are respectively shown in Fig. 7 and Fig. 8. The equivalent analytical loop

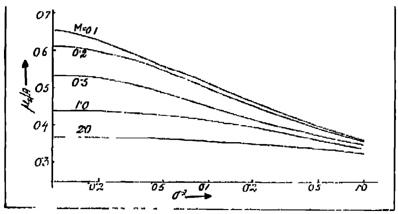

Fig. 7. Variation of the equivalent linearised gain of the phase detector for the signal with the variance of the noise at the input to the phase detector for different values of the modulation error (M).

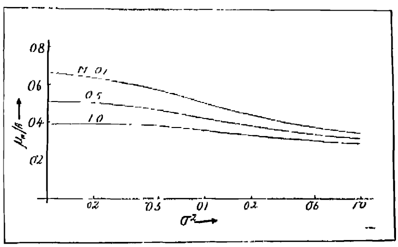

Fig. 8. Variation of the equivalent linearised gain of the phase detector for the noise with the variance of the noise at the input to the phase detector for different values of the modulation error (M).

have already been shown in Fig. 3(a) and Fig. 3(b) Now it is a simple matter to show that the modulation error M is given by

$$M = \frac{m_i}{\left|1+K\,\dfrac{F(jw)}{(jw)}\right|}, \qquad \ldots \quad (6.6)$$

wherz

$$K_S = \beta \mu_S.$$

Denoting the power spectral density of $N(p)$ as $G(w)$ (see section 2), the variance of the noise at the input to the phase detector is given by

$$\sigma^2 = \frac{1}{4\pi A^2} \int_{-\infty}^{+\infty} \left| \frac{KF(jw)}{jw+K_N F(jw)} \right|^2 G_1(w)dw), \quad \ldots \quad (6.7)$$

where

$$K_N = \beta_{\mu N}.$$

Therefore the modulation index at the output of the VCO is given by

$$m_0 = m_i \left| \frac{K\, F(jw)}{jw+K\, F(jw)} \right|. \quad \ldots \quad (6.8)$$

Analytical computation of the output SNR in terms of the input SNR is rather difficult because equivalent linearised gains depend both on σ^2 and M in a way given by Eq. (6.4) and Eq. (6.5) and again σ^2 and M depend on σ_i^2 and m_i in a Eq. (6.6) and Eq. (6.7). Therefore it is advisable to employ graphical methods for computation. By way of illustration one may take the example of the APC circuit with a simple lag filter of the form .

$$F(jw) = \frac{1}{1+jwT} \quad \ldots \quad (6.9)$$

where

$$\sigma^2 \frac{A^2}{\sigma_i^2} = (1+w_i T) \Big/ \left\{ \left[.04+.32(\sigma)^{-1/3}{}_1F_1\left(\frac{1}{6}, 1, -\frac{M^2}{2\sigma^2}\right) \right] \left[.04 + .32(\sigma)^{-1/3}{}_1F_1\left(1/6, 2- \frac{M^2}{2\sigma^2}\right) + \frac{\omega_i}{K}(1+w_i T) \right\} \right] \quad (6.10)$$

and

$$M^2 = \frac{m_i w^2}{K\left(.04+.7(\sigma)^{-1/3}\, {}_1F_1\left(\frac{1}{6}, 2, -\frac{M^2}{2\sigma^2}\right)\right\} - w^2T^2 \Big]^2 + w^2} \quad \ldots \quad (6.11)$$

Now taking $w_i/K = 5$, $w_i T = 1$, $K = 2\times 400$ rad/sec and $W = 2.0\times 400$ rad/sec one can easilyshow that Eq. (6 11) reduce to

$$\sigma^2 = \frac{A^2}{\sigma_i^2} \frac{2}{\left[04+.32(\sigma)^{-1/3}\, {}_1F_1\left(1/6, 1, -\frac{M^2}{2\sigma^2}\right)\right] \left[.04+.32(\sigma)^{-1/3}{}_1F_1\left(1/6, 2, \frac{M^2}{2\sigma^2}\right)\right]_{,10}} \quad \ldots \quad (6.10a)$$

and

$$M = \frac{m_i}{\left[\left\{.04 + .32(\sigma)^{-1/3}\, {}_1F_1\left(1/6, 2, -\frac{M^2}{2\sigma^2}\right) - 2\right\} + 1\right]^{\frac{1}{2}}} \tag{6.11a}$$

The plots of Eq. (6.10a) and Eq. (6.11a) are shown respectively in Fig 9a and Fig. 9b. From these plots it is easy to find the value of σ_i^2 and m_i^2 for a particular value of σ^2 and M^2 and hence the value of the output SNR can be found out in terms of the input SNR.

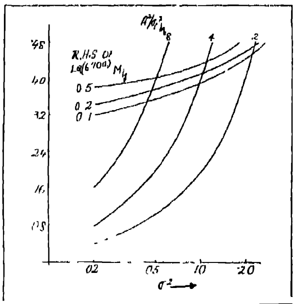

Fig. 9(a) Plots of Eq. (6.10a) and Eq. (6.11a). A comparison of Fig 9(a)

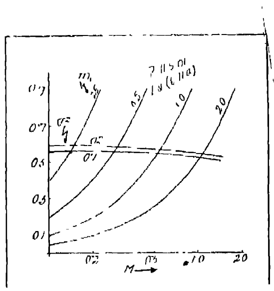

Fig 9(b) With the plot of Fig. 9(b) yields the values of noise error and the modulation eror in terms of variance of input noise and input modulation index.

The analysis of the off-tuned case is not, however, difficult. In this case one will have to take into account the steady state phase difference between the VCO and the incoming signal and it is quite logical to expect that the gain of the phase detector with respect to the signal will be smaller than the in-tuned case. Therefore the probability of loss of lock per cycle for a signal with a definite modulation index in the off-tuned case will be correspondingly larger than the in-tuned case.

Let us now consider the effect of the variation of the gain parameter 'K' with input signal to noise ratio on the system performance. From linear feedback theory it is known that the closed loop bandwidth is always larger than the open loop bandwith. The closed loop bandwidth is again a function of the gain parameter 'K'. To understand the effect of the variation of the gain parameter on the system performance we may consider what happens if K is increased from a nominal value K_0. An increase in K results in a smaller modulation phase error and a large modulation loop bandwidth. The proportion of the noise phase output to signal phase output will now be smaller than before although the loop bandwidth has increased. This will be clear if we compare the output phase contriutions due to message and noise if their spectral densities in the bandwidth under consideration are the same. In such a case the phase contribution due to noise the will be $1/K$ that due to the message

The reduction factor will not be as large as the ratio of the gains because the noise power accepted will now be larger because of increase of noise bandwidth Note that in this case the closed loop bandwidth will have to be kept larger than the message bandwidth to ensure adequate tracking capability

From the above discussion it appears that it will possible be helpful to limit the maximum value of the input to the VCO An alternative approach is to control the filter parameters in such a way that the closed loop bandwidth does not vary significantly with signal strength.

Noise-Bandwidth and Root-mean-square Phase Error

In this section the concepts of noise-bandwidth and root-mean-square phase error of an automatic phase control circuit will be briefly reviewed with a view to designing an optimum system The concepts of noise bandwidth and r.m.s phase error are important in studying the performance of an automatic phase control circuit particularly when it is tracking a signal that is contaminated with uncontrolled disturbances. The performance of an automatic phase control circuit in tracking a noisy signal will be judged best when the output phase of the voltage controlled oscillator faithfully follows that input phase variations and at the same time ignores the uncontrolled disturbances as for as possible

When the APC circuit is in lock with the input signal and is tracking a noisy signal it is reasonable, although not very accurate assumption, to consider the phase detector output to be linearly dependent on the phase difference between the signals at the input to the phase detector. This gives the linearised version of the APC circuit.

The concept of loop noise-bandwidth will enable one to have an useful information regarding the propagation of noise through an APC circuit and even to design the required from of the loop filter which will result in Wiener optimum linear system. The loop noise-bandwidth can be defined as

$$B_n = \int^{\infty} G(jw)\ ^2 df \qquad \ldots \quad (7.1)$$

where $G(jw)$ is the normalised closed loop transfer function of the linearised model from the output phase to the input phase of the voltage controlled oscillator. Eq. (7.1) can also be written as

$$B_n = \frac{1}{4\pi j} \int_{-j\infty}^{j\infty} G(p)\, G(-p) dp \qquad \ldots \quad (7.2)$$

Thus B_n is the bandwidth of an ideal square cut-off low-pass filter which produces the same amount of noise power output as does the linear system with transfer function $G(p)$. It is to be noted that the very basis of this definition depends on the assumption that noise power spectral density at the input to the linearised model is constant over all frequencies (white noise) White noise cannlt occur in practice. The more realistic approach is to consider the bandlimited white noise at the input to the linearised system and the more useful definition for the loop noise bandwidth is given

$$B_L = \frac{1}{4\pi j} \int_0^{jw_c} G(p)\, G(-p) dp \qquad \dots \quad (7.3)$$

where $w_c/2\pi$ is the cut-off frequency of the input filter to the APC system. Correspondingly the r m s. phase error or the variance of the noise at the input to the phase detector can be defined as

$$\sigma^2 = \frac{1}{4\pi A^2} \int_{-\infty}^{\infty} \left| \frac{KFpw)}{jw + KF(jw)} \right|^2 G_j(w) dw \qquad \dots \quad (7.4)$$

where the symbols have their usual significance as stated elsewhere in the text. The limits of the integration as stated earlier for the most parctical case should be taken over the input bandwidth. But if the input bandwidth is large compared to the close loop bandwidth then the limits of integration can be taken from $-\infty$ to $+\infty$ without introducing much error to the computed value.

For purpose of comparison the expressions for the noise bandwidth of the APC circuit with the simple lag filter (see Eq. (3.10) of section 3) obtained from the Eqs. (6 2) and (6.3) are given below :

$$B_n = K/4, \qquad \dots \quad (7.5)$$

$$B_L = \frac{K}{4} \left[\frac{1}{2\pi\sqrt{2KT-1}} \log_e \left\{ \frac{\left(\frac{w_c}{K}\right)^2 - \frac{\sqrt{4KT-1}}{KT}\left(\frac{w_c}{K}\right) + \frac{1}{KT}}{\left(\frac{w_c}{K}\right)^2 + \frac{\sqrt{4KT-1}}{KT}\left(\frac{w_c}{K}\right) + \frac{1}{KT}} \right\} + \frac{1}{4} \tan^{-1} \left(\frac{\frac{w_c}{K} \cdot \frac{1}{KT}}{\frac{1}{KT} - \left(\frac{w_c}{K}\right)^2} \right) \right] \qquad \dots \quad (7\ 6)$$

From the above expression it is seen that B_L tends to B_n as w_c tends to an infinitely large value.

For the second order loop, i.e. taking the filter transfer function of the form

$$F(P) = \frac{1+xpT}{1+(1+x)PT} \tag{7.7}$$

the expressions for the noise bandwidth and m s. phase error are respectively given by

$$B_n = \frac{K}{4} \cdot \frac{[1+x(1+xKT)]}{(1+x)(1+xKT)} \tag{7.8}$$

$$\sigma^2 = \frac{\sigma_i^2}{A^2} \cdot \frac{(1+x)KT+(1+xKT)\dfrac{K}{w_i}+(xKT)^2}{(1+xKT)\left[(1+x)\dfrac{w_i}{K}(KT)+(1+xKT)+\dfrac{K}{w_i}(1+xKT)\right]} \quad \ldots \tag{7.9}$$

The plots of B_n and σ^2 are shown in Fig. 10.

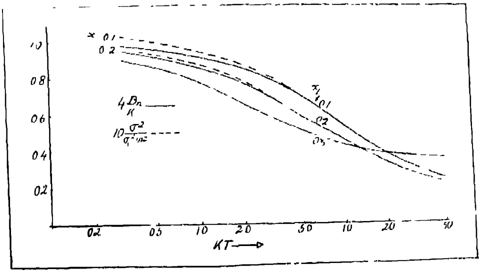

Fig. 10 Variation of the close-loop noise bandwidth and the root-mean-square phase error of an automatic phase control circuit with KT

Experimental Set-up and results

Fig. 11 shows the experimental set-up for making measurements on the response of the APC circuit to a F.M. signal and the CW signal contaminated with band-limited stationary random noise The bandlimited noise has been obtained

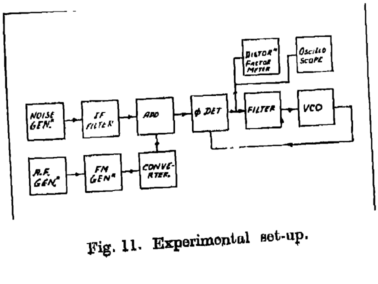

Fig. 11. Experimental set-up.

by passing white Gaussian noise through an I-F filter with the centre frequency of 500 Kc/s Incidentally it is to be noted that the centre frequency of the IF filter must be equal to the centre frequency of the voltage controlled oscillator. The detailed circuit diagram of the APC circuit has been given in the reference (Chakrabarti, N. B. and Biswas, B N., 1964). The input amplifier feeding the phase detector should have a flat top response. Presence of dip anywhere in the characteristics, is likely to produce spurious effect and sometimes a type of oscillations (Biswas, B. N. 1964).

Experimental results regarding the performance of the APC circuit in relation to the reception of a FM-signal with different values of input SNR are shown in Figs. 12(a) and 12(b). Fig. 12(a) shows the capability of the APC circuit in

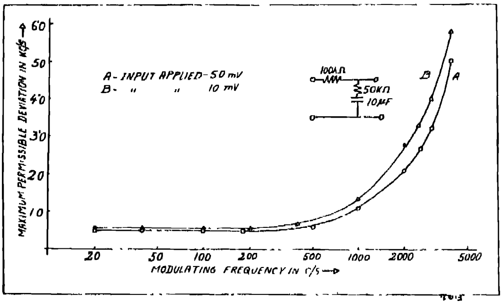

Fig.12(a) Experimental observations regarding the performance of the automatic phase con-

handling a FM signal with maximum permissible input modulation index at different values of the modulating frequency. The variation of the output noise level of the APC system with the input noise level when the system is tracking a low index FM signal contaminated with stationary random noise is shown in Fig 12(b) These experimental results are in good agreement with the results of the analysis.

CONCLUSIONS

In this paper the response of the automatic phase control circuit to a CW signal contaminated with stationary random noise and a FM signal contaminated with random noise has been studied, and the performance of the APC circuit with respect to such signals has also been studied experimentally. It has been found that the conclusions of the analysis presented in the text give a resonable estimate regarding the behaviour of the APC circuit with respect to signals corrupted with random noise The response of an APC circuit preceded by a limiter to a FM signal contaminated with random noise will be taken up in a future communication,

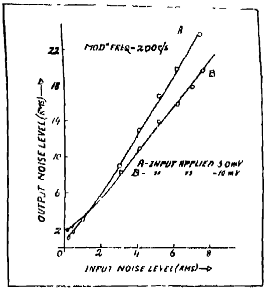

Fig.12(b) trol circuit in respect of reception of a FM signal. Fig 12(a) shows the variation of maximum permissible value of the input modulation index with modulating frequency and Fig. 12(b) shows the variation of the output noise level with input noise level when the APC circuit is tracking a low index FM signal contaminated with stationary random noise.

ACKNOWLEDGMENT

The author takes the opportunity of expressing his gratitude to Prof N B Chakrabarti of the Institute of Radio Physics and Electronics, Calcutta (now at the Indian Institute of Technology, Kharagpur), for suggesting the problem and supervision of the work described The author is indebted to Prof J N. Bhar, D.Sc., F.N I. of the Institute of Radio Physics and Electronics, Calcutta for his kind interest and providing the author with all the research facilities The author wishes to thank Prof A. Mukherjee, D.Sc , of the University of Burdwan, for his kind interest and encouragement. The author also wishes to thank Mr P.L Dhar Bhowmick for helpful discussions. Thanks are also due to Messrs A N Chakravarti, A. K. Datta and S. P Nag for assistance in the experimental work

APPENDIX

A.1. *Derivation of the Governing of the APC Loop when the Incoming Signal is Contaminated with Stationary Random Noise.*

Let us consider the APC circuit as shown in Fig 1 The input to the system has been assumed to be of the form :

$$e_i(t) = A \sin (w_1 t + m_i \sin wt) + X(t) \sin w_2 t + Y(t) \cos w_2 t \qquad \ldots \quad (A\ 1)$$

where symbols have their usual significance as stated elsewhere in the text. The output of the voltage controlled oscillator is assumed to be of the form :

$$e(\phi)(t) = 2 \cos (w_2 t + m_0 \sin wt + \epsilon(t)) \qquad \ldots \quad (A.2)$$

Therefore, the output of the phase detector, which is a multiplicative device, is given by

$$e\phi(t) = A \sin[(w_1t+w_2)t+(m_i+m_0)\sin wt+\epsilon(t)]$$
$$+A\sin[(w_1-w_2)t+(m_i-m_0)\sin wt-\epsilon(t)]$$
$$+X(t)\sin[2w_2t+m_i\sin wt+\epsilon(t)]+Y(t)\cos[2w_2t+m_i\sin wt+\epsilon(t)]$$
$$-X(t)\sin[m_0\sin wt+\epsilon(t)]+Y(t)\cos[m_0\sin wt+\epsilon(t)] \quad \dots \quad (A.3)$$

Since a phase detector is followed by a low pass filter one can easily neglect the high frequency term in Eq. (A.3) and can write the actual output of the phase detector as

$$l_p(t) = A\sin[(w_1-w_2)t+(m_i-m_0)\sin wt-\epsilon(t)]$$
$$-X(t)\sin[m_0\sin wt+\epsilon(t)]+Y(t)\cos[m_0\sin wt+\epsilon(t)] \quad \dots \quad (A.4)$$

Therefore the governing equation of the APC loop is given by

$$\frac{d\phi}{dt} = \Omega-\beta F(p)[A\sin\phi+N(t)]+\frac{d}{dt}(m_i\sin wt) \quad \dots \quad (A.5)$$

where the symbols have their usual significance as stated elsewhere.

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